International Journal of Theoretical Physics, Vol. 39, No. 9, 2000

Braided Covariance of the Braided Differential Bialgebras under Quantized Braided Groups

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Received November 22, 1999

The braided differential bialgebras on braided matrix algebras (with both multiplicative and additive coproducts) and on quantum hyperplanes (with additive coproduct) are proven to be covariant under the braided coactions of the quantized braided groups, which contain the usual quantum group-covariance as a special case. This means that the above braided differential bialgebras have more and richer symmetries. It is also shown that the braided matrix algebra itself and the related braided differential algebra constitute two braided rings with the two above-mentioned coproducts.

1. INTRODUCTION

In recent years there has been a great deal of interest in quantum and braided differential algebras due to their importance in mathematical physics. Some quantum (braided) differential bialgebras were studied by many authors (e.g., Woronowicz, 1989; Brzezinski, 1993; Wess and Zumino, 1990; Baez, 1991; Iseav and Vladimirov, 1995; Vladimirov, 1994; Schlieker and Zumino, 1995; Drabant, 1997) and their covariance with respect to the (co)action of certain quantum groups was discussed (e.g., Iseav and Vladimirov, 1995; Wess and Zumino, 1990; Baez, 1991). On the other hand, a kind of more general algebraic structure called a quantized braided group (QBG) was proposed more recently by Hlavaty (1994, 1997) and some related algebras were also investigated (Gao and Gui, 1997; Hlavaty, 1994).

In this paper, we extend these discussions and show that the braided *differential* bialgebras on braided matrix algebras (with both additive and multiplicative coproducts) and on quantum hyperplanes (with an additive

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0020-7748/00/0900-2179$18.00/0

coproduct) are also covariant with respect to the *braided* coaction of the QBG, which contains the coactions of the quantum (unbraided) group and braided (unquantized) group as two special cases. These mean that the above-mentioned braided differential bialgebras have more and richer symmetries. Moreover, we also prove that the braided differential bialgebras on the braided matrix algebras with additive and multiplicative coproducts have the so-called braided ring structure.

For convenience, in this paper we use the R-matrix formalism (Faddeev *et al.*, 1990) and suppose the matrix R is of Hecke type

$$PRPR = \lambda PR + 1, \qquad \lambda = q - q^{-1} \tag{1.1}$$

where P is the permutation matrix and q is the quantun deformation parameter.

In Section 2, we recall briefly the notations of the QBG and some related braided linear algebras. Section 3 proves the covariance of the braided differential bialgebras under the braided coaction of QBG. The braided ring structures of the braided differential bialgebrs on the braided matrix algebras are shown in Section 4. Section 5 is devoted to some conclusions and discussions.

2. QUANTIZED BRAIDED GROUP

For later use, here we recall some related notations and properties of the QBG (Hlavaty, 1994). Let $T = \{T^i_j\}^N_{i,j=1}$ be a matrix of N^2 elements T^i_j and $R, Z \in M_N \otimes M_N$ be a solution of the following set of quantum Yang–Baxter-type equations:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \qquad Z_{12}Z_{13}Z_{23} = Z_{23}Z_{13}Z_{12} \tag{2.1}$$
$$R_{12}Z_{13}Z_{23} = Z_{23}Z_{13}R_{12}, \qquad Z_{12}Z_{13}R_{23} = R_{23}Z_{13}Z_{12}$$

Then the quantized braided (matrix) group $A(R, Z)$ is defined as follows:

(i) $A(R, Z)$ is a bialgebra generated by $\{T^i_j\}$ and 1 with the relations

$$R_{12}Z^{-1}_{12}T_1Z_{12}T_2 = Z^{-1}_{21}T_2Z_{21}T_1R_{12} \tag{2.2}$$

$$\Delta(T^i_j) = T^i_k \otimes T^k_j, \qquad \varepsilon(T^i_j) = \delta^i_j \tag{2.3}$$

and the braidings

$$Z^{-1}_{12}T'_1Z_{12}T_2 = T_2Z^{-1}_{12}T'_1Z_{12} \tag{2.4}$$

where Δ and ε are coproduct and counit, and for the braiding relation we have used the notation $a \otimes 1 \equiv a$, $1 \otimes a \equiv a'$ for any algebraic element a and $a'b = \Psi(a \otimes b)$.

(ii) There is an antipode S obeying the axioms

$$S(T)T = TS(T) = I, \qquad S(1) = 1 \tag{2.5}$$

In the follwing we write $S(T)$ as T^{-1}.

If $A(R, Z)$ satisfies the condition (i) only, we call it a quantized braided (matrix) bialgebra.

For the cases $Z = I$ or $Z = R$, the QBG $A(R, Z)$ reduces to the usual quantum (matrix) group $A(R)$ (Manin, 1988; Faddeev *et al.*, 1990; Majid, 1990) or braided (matrix) group $B(R)$ (Majid, 1991, 1993), respectively. We have found the covariance of $B(R)$ (as an algebra) under the braided coaction of $A(R, Z)$ (Gao and Gui, 1997). That the quantum covector space $V^*(R)$ [resp., vector space $V(R)$] generated by $\{1, x_i\}$ (resp. $\{1, v_j\}$) with relation $q\mathrm{x}_1\mathrm{x}_2 = \mathrm{x}_2\mathrm{x}_1R_{12}$ (resp., $q\mathrm{v}_1\mathrm{v}_2 = R_{12}\mathrm{v}_2\mathrm{v}_1$) is covariant to the QBG has also been pointed out by Hlavaty (1994). In the next section, we shall extend these discussions to the *differential bialgebras* on $B(R)$ and $V^*(R)$, etc.

3. *A*(*R*, *Z*) COVARIANCE OF THE BRAIDED DIFFERENTIAL BIALGEBRAS

We first recall that the differential complex on $B(R)$ is generated by $\{1, U^i_j, \mathrm{d}U^i_j\}$ with relations (Iseav and Vladimirov, 1995; Vladimirov, 1994)

$$R_{21}U_1R_{12}U_2 = U_2R_{21}U_1R_{12} \tag{3.1a}$$

$$R_{21}U_1R_{12}\,\mathrm{d}U_2 = \mathrm{d}U_2\,R_{21}U_1R_{21}^{-1} \tag{3.1b}$$

$$R_{21}\,\mathrm{d}U_1\,R_{12}\,\mathrm{d}U_2 = -\mathrm{d}U_2\,R_{21}\,\mathrm{d}U_1\,R_{21}^{-1} \tag{3.1c}$$

and the algebra (3.1) admits two coproducts. One of them is multiplicative,

$$\overline{\Delta}(U^i_j) = U^i_k \otimes U^k_j, \qquad \text{e.g.,} \quad \overline{\Delta}U = U \otimes U \equiv UU', \qquad \overline{\varepsilon}(U) = \mathbf{1} \tag{3.2a}$$

$$\overline{\Delta}(\mathrm{d}U) = \mathrm{d}U \otimes U + U \otimes \mathrm{d}U \equiv \mathrm{d}UU' + U\mathrm{d}U', \qquad \overline{\varepsilon}(\mathrm{d}U) = 0 \tag{3.2b}$$

with the braiding relations

$$R_{12}^{-1}U'_1R_{12}U_2 = U_2R_{12}^{-1}U'_1R_{12} \tag{3.3a}$$

$$R_{12}^{-1}\,\mathrm{d}U'_1\,R_{12}U_2 = U_2R_{12}^{-1}\,\mathrm{d}U'_1\,R_{12} \tag{3.3b}$$

$$R_{12}^{-1}U'_1R_{12}\,\mathrm{d}U_2 = \mathrm{d}U_2\,R_{12}^{-1}U'_1R_{12} \tag{3.3c}$$

$$R_{12}^{-1}\,\mathrm{d}U'_1\,R_{12}\,\mathrm{d}U_2 = -\mathrm{d}U_2\,R_{12}^{-1}\,\mathrm{d}U'_1\,R_{12} \tag{3.3d}$$

The other coproduct is additive,

$$\underline{\Delta}U = U \otimes 1 + 1 \otimes U \equiv U + \tilde{U}, \qquad \underline{\varepsilon}(U) = 0 \tag{3.4a}$$

$$\underline{\Delta}(\mathrm{d}U) = \mathrm{d}U \otimes 1 + 1 \otimes \mathrm{d}U \equiv \mathrm{d}U + \mathrm{d}\tilde{U}, \qquad \underline{\varepsilon}(\mathrm{d}U) = 0 \tag{3.4b}$$

with the braiding relations

$$R_{21}\tilde{U}_1 R_{12} U_2 = U_2 R_{21} \tilde{U}_1 R_{21}^{-1} \tag{3.5a}$$

$$R_{21}\tilde{U}_1 R_{12}\, dU_2 = dU_2\, R_{21}\tilde{U}_1 R_{21}^{-1} - \lambda P_{12} U_1 R_{12}\, d\tilde{U}_2 \tag{3.5b}$$

$$R_{21}\, d\tilde{U}_1\, R_{12} U_2 = U_2 R_{21}\, d\tilde{U}_1\, R_{12} \tag{3.5c}$$

$$R_{21}\, d\tilde{U}_1\, R_{12}\, dU_2 = -dU_2\, R_{21}\, d\tilde{U}_1\, R_{12} \tag{3.5d}$$

Here, to distinguish the two coproducts and the related operations, we have used the different symbols $\overline{\Delta}$, $\underline{\Delta}$; U', $\tilde{U}$; $\overline{\varepsilon}$, $\underline{\varepsilon}$ etc., which remind us that we are computing in different coalgebraic structures.

The differential algebra (3.1) will be denoted by Ω_B; we also denote the differential bialgebra defined by (3.1)–(3.3) as $\overline{\Omega}_B$ and that defined by (3.1), (3.4), (3.5) as $\underline{\Omega}_B$, i.e., $\overline{\Omega}_B \equiv (\Omega_B, \overline{\Delta}, \overline{\varepsilon})$, $\underline{\Omega}_B \equiv (\Omega_B, \underline{\Delta}, \underline{\varepsilon})$. Moreover for later use we mention that on $\underline{\Omega}_B$ the antipode $\underline{S}$ can be introduced as

$$\underline{S}(U) = -U, \qquad \underline{S}(dU) = -dU \tag{3.6a, b}$$

such that $\underline{\Omega}_B$ becomes a braided Hopf algebra (Iseav and Vladimirov, 1995).

Theorem 1. If $A(R, Z)$ is a QBG as defined in Section 1 and R is of Hecke type as in (1.1), then the braided differential bialgebras $\overline{\Omega}_B$ and $\underline{\Omega}_B$ both are covariant under the braided (rigid) coaction of $A(R, Z)$,

$$\beta\colon \quad U \mapsto \beta(U) = T^{-1}UT, \quad dU \mapsto \beta(dU) = T^{-1}\, dUT \tag{3.7}$$

with the following braiding relations:

$$Z_{12}^{-1} T_1' Z_{12} U_2 = U_2 Z_{12}^{-1} T_1' Z_{12}, \qquad Z_{12}^{-1} T_1' Z_{12}\, dU_2 = dU_2\, Z_{12}^{-1} T_1' Z_{12} \tag{3.8}$$

Proof. As explained in Majid (1993) and Gao and Gui (1997), the notations $T^{-1}UT$, $T^{-1}\, dUT$ in (3.7) mean precisely $T^{-1\prime}UT'$, $T^{-1\prime}\, dUT'$ by definition and because T and U, dU live in different algebras, there is no danger of confusing braidings and inverse braidings. Thus, for simplicity, we suppress the primes on T^{-1} and T in the following calculations.

The $A(R, Z)$ covariance of the algebraic relation (3.1a) has been verified by us (Gao and Gui, 1997) in a more general form. Here we consider (3.1b) and (3.1c). For (3.1b), from (2.2), (2.5), (3.1), and (3.8) we have

$$\begin{aligned}
R_{21}\beta(U_1)R_{12}\beta(dU_2) &= R_{21}T_1^{-1}U_1\underline{T_1R_{12}T_2^{-1}}\, dU_2\, T_2 \\
&= R_{21}T_1^{-1}\underline{U_1Z_{21}^{-1}T_2^{-1}Z_{21}}R_{12}\underline{Z_{12}^{-1}T_1Z_{12}\, dU_2}T_2 \\
&= \underline{R_{21}T_1^{-1}Z_{21}^{-1}T_2^{-1}Z_{21}}U_1R_{12}\, dU_2\, Z_{12}^{-1}T_1Z_{12}T_2 \\
&= T_2^{-1}Z_{12}^{-1}T_1^{-1}Z_{12}\underline{R_{21}U_1R_{12}\, dU_2}\, Z_{12}^{-1}T_1Z_{12}T_2 \\
&= T_2^{-1}\underline{Z_{12}^{-1}T_1^{-1}Z_{12}\, dU_2}\, R_{21}U_1\underline{R_{21}^{-1}Z_{12}^{-1}T_1Z_{12}T_2}
\end{aligned}$$

$$= T_2^{-1}\,dU_2\;Z_{12}^{-1}T_1^{-1}Z_{12}R_{21}\underline{U_1Z_{21}^{-1}T_2Z_{21}}T_1R_{21}^{-1}$$

$$= T_2^{-1}\,dU_2\;\underline{Z_{12}^{-1}T_1^{-1}Z_{12}R_{21}Z_{21}^{-1}T_2Z_{21}}U_1T_1R_{21}^{-1}$$

$$= T_2^{-1}\,dU_2\;T_2R_{21}T_1^{-1}U_1T_1R_{21}^{-1}$$

$$= \beta(dU_2)R_{21}\beta(U_1)R_{21}^{-1}$$

where the underlines indicate the parts to which the next operations are to be applied. Similarly, for (3.1c) we can prove that

$$R_{21}\beta(dU_1)R_{12}\beta(dU_2) = -\beta(dU_2)R_{21}\beta(dU_1)R_{21}^{-1}$$

In addition, it can also be readily shown that the braided comodule coalgebra condition

$$(\Delta \otimes 1)\beta = (1 \otimes 1 \otimes \cdot)(1 \otimes \Psi \otimes 1)(\beta \otimes \beta)\Delta \tag{3.9}$$

is satisfied for both $\overline{\Omega}_B$ and $\underline{\Omega}_B$ (corresponding to $\Delta = \overline{\Delta}$ and $\underline{\Delta}$, respectively) and all braidings in (3.3) and (3.5) are consistent with the braided coaction (3.7) of $A(R, Z)$. As examples, we examine the braidings (3.3b) and (3.5b). Noting the property of the coaction on tensor products, for (3.3b), from (2.1), (2.5), (3.7), and (3.8) we have

$$\begin{aligned}
R_{12}^{-1}\beta(dU_1')R_{12}\beta(U_2) &= R_{12}^{-1}T_1^{-1}\,dU_1'\;\underline{T_1R_{12}T_2^{-1}}U_2T_2 \\
&= R_{12}^{-1}T_1^{-1}\;\underline{dU_1'\;Z_{21}^{-1}T_2^{-1}Z_{21}}R_{12}\underline{Z_{12}^{-1}T_1Z_{12}U_2}T_2 \\
&= \underline{R_{12}^{-1}T_1^{-1}Z_{21}^{-1}T_2^{-1}Z_{21}}\;dU_1'\;R_{12}U_2Z_{12}^{-1}T_1Z_{12}T_2 \\
&= T_2^{-1}Z_{12}^{-1}T_1^{-1}Z_{12}\underline{R_{12}^{-1}\;dU_1'\;R_{12}U_2}Z_{12}^{-1}T_1Z_{12}T_2 \\
&= T_2^{-1}\underline{Z_{12}^{-1}T_1^{-1}Z_{12}U_2}R_{12}^{-1}\;dU_1'\;\underline{R_{12}Z_{12}^{-1}T_1Z_{12}T_2} \\
&= T_2^{-1}U_2Z_{12}^{-1}T_1^{-1}Z_{12}R_{12}^{-1}\;\underline{dU_1'\;Z_{21}^{-1}T_2Z_{21}}T_1R_{12} \\
&= T_2^{-1}U_2\underline{Z_{12}^{-1}T_1^{-1}Z_{12}R_{12}^{-1}Z_{21}^{-1}T_2Z_{21}}\;dU_1'\;T_1R_{12} \\
&= T_2^{-1}U_2T_2R_{12}^{-1}T_1^{-1}\;dU_1'\;T_1R_{12} \\
&= \beta(U_2)R_{12}^{-1}\beta(dU_1')R_{12}
\end{aligned}$$

Similarly, for (3.5b) we have

$$
\begin{aligned}
&R_{21}\beta(\tilde{U}_1)R_{12}\beta(\mathrm{d}U_2)\\
&\quad = R_{21}T_1^{-1}\tilde{U}_1\underline{T_1R_{12}T_2^{-1}}\,\mathrm{d}U_2\,T_2\\
&\quad = R_{21}T_1^{-1}\underline{\tilde{U}_1Z_{21}^{-1}T_2^{-1}Z_{21}}R_{12}\underline{Z_{12}^{-1}T_1Z_{12}\,\mathrm{d}U_2}\,T_2\\
&\quad = \underline{R_{21}T_1^{-1}Z_{21}^{-1}T_2^{-1}Z_{21}}\tilde{U}_1R_{12}\,\mathrm{d}U_2\,Z_{12}^{-1}T_1Z_{12}T_2\\
&\quad = T_2^{-1}Z_{12}^{-1}T_1^{-1}Z_{12}\underline{R_{21}\tilde{U}_1R_{12}\,\mathrm{d}U_2}\,Z_{12}^{-1}T_1Z_{12}T_2\\
&\quad = T_2^{-1}\underline{Z_{12}^{-1}T_1^{-1}Z_{12}\,\mathrm{d}U_2}\,R_{21}\tilde{U}_1\underline{R_{21}^{-1}Z_{12}^{-1}T_1Z_{12}T_2}\\
&\qquad - \lambda\underline{T_2^{-1}Z_{12}^{-1}T_1^{-1}Z_{12}P_{12}}U_1R_{12}\,\underline{\mathrm{d}\tilde{U}_2\,Z_{12}^{-1}T_1Z_{12}}T_2\\
&\quad = T_2^{-1}\,\mathrm{d}U_2\,Z_{12}^{-1}T_1^{-1}Z_{12}R_{21}\underline{\tilde{U}_1Z_{21}^{-1}T_2Z_{21}}T_1R_{21}^{-1}\\
&\qquad - \lambda P_{12}T_1^{-1}\underline{Z_{21}^{-1}T_2^{-1}Z_{21}U_1}R_{12}Z_{12}^{-1}T_1Z_{12}\,\mathrm{d}\tilde{U}_2\,T_2\\
&\quad = T_2^{-1}\,\mathrm{d}U_2\,\underline{Z_{12}^{-1}T_1^{-1}Z_{12}R_{21}Z_{21}^{-1}T_2Z_{21}}\tilde{U}_1T_1R_{21}^{-1}\\
&\qquad - \lambda P_{12}T_1^{-1}U_1\underline{Z_{21}^{-1}T_2^{-1}Z_{21}R_{12}Z_{12}^{-1}T_1Z_{12}}\,\mathrm{d}\tilde{U}_2\,T_2\\
&\quad = T_2^{-1}\,\mathrm{d}U_2\,T_2R_{21}T_1^{-1}\tilde{U}_1T_1R_{21}^{-1} - \lambda P_{12}T_1^{-1}U_1T_1R_{12}T_2^{-1}\,\mathrm{d}\tilde{U}_2\,T_2\\
&\quad = \beta(\mathrm{d}U_2)R_{21}\beta(\tilde{U}_1)R_{21}^{-1} - \lambda P_{12}\beta(U_1)R_{12}\beta(\mathrm{d}\tilde{U}_2)
\end{aligned}
$$

Hence we see that both $\overline{\Omega}_B$ and $\underline{\Omega}_B$ are braided $A(R, Z)$-comodule bialgebras and the theorem is proved. ■

For quantum spaces, as an example, we consider the case on $V^*(R)$ [the cases for $V(R)$ and fermionic hyperplanes are entirely similar]. The braided differential bialgebra on $V^*(R)$, denoted by Ω_{V^*}, is generated by $\{1, x_i, \mathrm{d}x_i\}$ with the relations

$$q\mathrm{x}_1\mathrm{x}_2 = \mathrm{x}_2\mathrm{x}_1R_{12} \tag{3.10a}$$

$$\mathrm{x}_1\,\mathrm{dx}_2 = q\,\mathrm{dx}_2\,\mathrm{x}_1R_{12} \tag{3.10b}$$

$$\mathrm{dx}_1\,\mathrm{dx}_2 = -q\,\mathrm{dx}_2\,\mathrm{dx}_1\,R_{12} \tag{3.10c}$$

$$\tilde{\Delta}(\mathrm{x}) = \mathrm{x}\otimes 1 + 1\otimes \mathrm{x} = \mathrm{x} + \mathrm{x}', \qquad \tilde{\varepsilon}(\mathrm{x}) = 0 \tag{3.11a}$$

$$\tilde{\Delta}(\mathrm{dx}) = \mathrm{dx}\otimes 1 + 1\otimes \mathrm{dx} = \mathrm{dx} + \mathrm{dx}', \qquad \tilde{\varepsilon}(\mathrm{dx}) = 0 \tag{3.11b}$$

and the braidings

$$\mathrm{x}_1'\mathrm{x}_2 = q\mathrm{x}_2\mathrm{x}_1'R_{12}, \qquad q^{-1}\mathrm{x}_1'\,\mathrm{dx}_2 = \mathrm{dx}_2\,\mathrm{x}_1'R_{12} + \lambda\mathrm{x}_1\,\mathrm{dx}_2' \tag{3.12a, b}$$

$$\mathrm{dx}_1'\,\mathrm{x}_2R_{21} = q\mathrm{x}_2\,\mathrm{dx}_1', \qquad \mathrm{dx}_1'\,\mathrm{dx}_2'\,R_{21} = -q\,\mathrm{dx}_2\,\mathrm{dx}_1' \tag{3.12c, d}$$

Theorem 2. Let $A(R, Z)$ be as in Theorem 1; then the braided differential bialgebra Ω_{V^*} is covariant with respect to the braided (rigid) coaction β: $\mathrm{x} \mapsto \mathrm{x}T$, $\mathrm{dx} \mapsto \mathrm{dx}T$ with the following braiding relations:

$$T'_1\mathrm{x}_2 = \mathrm{x}_2 Z_{12}^{-1} T'_1 Z_{12}, \qquad T'_1\,\mathrm{dx}_2 = \mathrm{dx}_2\, Z_{12}^{-1} T'_1 Z_{12} \tag{3.13}$$

Proof. The $A(R, Z)$ covariance of the algebraic relation (3.10a) has been pointed out by Hlavaty (1994) in a more general form; the covariance of (3.10b) is shown as follows: from (2.1) (2.5), and (3.13) we have

$$\begin{aligned}\beta(\mathrm{x}_1)\beta(\mathrm{dx}_2) &= \mathrm{x}_1\underline{T_1\,\mathrm{dx}_2}\,T_2 = \underline{\mathrm{x}_1\,\mathrm{dx}_2}\,Z_{12}^{-1}T_1Z_{12}T_2\\ &= q\,\mathrm{dx}_2\,\mathrm{x}_1\underline{R_{12}Z_{12}^{-1}T_1Z_{12}T_2} = q\,\mathrm{dx}_2\,\underline{\mathrm{x}_1Z_{21}^{-1}T_2Z_{21}}T_1R_{12}\\ &= q\,\mathrm{dx}_2\,T_2\mathrm{x}_1T_1R_{12} = q\beta(\mathrm{dx}_2)\beta(\mathrm{x}_1)R_{12}\end{aligned}$$

Similarly, for (3.10c) we have

$$\beta(\mathrm{dx}_1)\beta(\mathrm{dx}_2) = -q\beta(\mathrm{dx}_2)\beta(\mathrm{dx}_1)R_{12}$$

Moreover, Ω_{V^*} is also a braided $A(R, Z)$-comodule coalgebra since one can readily verify that the condition (3.9) is fulfilled [here $\Delta = \tilde{\Delta}$; see (3.11)] and the braidings in (3.12) are consistent with the braided coaction of $A(R, Z)$. As an example, for (3.12b), from (2.1), (2.5), and (3.13), we have

$$\begin{aligned}q^{-1}\beta(\mathrm{x}'_1)\beta(\mathrm{dx}_2) &= q^{-1}\mathrm{x}'_1\underline{T_1\,\mathrm{dx}_2}\,T_2 = \underline{q^{-1}\mathrm{x}'_1\,\mathrm{dx}_2}\,Z_{12}^{-1}T_1Z_{12}T_2\\ &= \mathrm{dx}_2\,\mathrm{x}'_1\underline{R_{12}Z_{12}^{-1}T_1Z_{12}T_2} + \lambda\mathrm{x}_1\,\underline{\mathrm{dx}'_2\,Z_{12}^{-1}T_1Z_{12}}T_2\\ &= \mathrm{dx}_2\,\underline{\mathrm{x}'_1Z_{21}^{-1}T_2Z_{21}}T_1R_{12} + \lambda\mathrm{x}_1T_1\,\mathrm{dx}'_2\,T_2\\ &= \mathrm{dx}_2\,T_2\mathrm{x}'_1T_1R_{12} + \lambda\mathrm{x}_1T_1\,\mathrm{dx}'_2T_2\\ &= \beta(\mathrm{dx}_2)\beta(\mathrm{x}'_1)R_{12} + \lambda\beta(\mathrm{x}_1)\beta(\mathrm{dx}'_2)\end{aligned}$$

These imply that Ω_{V^*} is a braided $A(R, Z)$-comodule bialgebra. ■

The relations (3.1b), (3.1c), (3.3), (3.5), and (3.12) are not unique (Iseav and Vladimirov, 1995; Vladimirov, 1994). The discussions for the remaining relations are completely parallel, so in this paper we only consider the above cases in detail.

4. BRAIDED RING STRUCTURE OF THE BRAIDED DIFFERENTIAL BIALGEBRA ON *B*(*R*)

As mentioned in Section 3, the braided (matrix) differential algebra (3.1) admits two coproducts (3.2)–(3.3) and (3.4)–(3.5) (Iseav and Vladimirov, 1995). Now we consider the connection between these coproducts.

Definition 1. A braided ring is a braided bialgebra $(B, \overline{\Delta}, \overline{\varepsilon})$ with a second braided Hopf algebra structure $(B, \underline{\Delta}, \underline{\varepsilon}, \underline{S})$, which obeys the codistributivity axioms

$$\begin{aligned}(\mathrm{id} \otimes \circ) \circ \overline{\Delta}_{B\otimes B} \circ \underline{\Delta} &= (\underline{\Delta} \otimes \mathrm{id}) \circ \overline{\Delta},\\ (\circ \otimes \mathrm{id}) \circ \overline{\Delta}_{B\otimes B} \circ \underline{\Delta} &= (\mathrm{id} \otimes \underline{\Delta}) \circ \overline{\Delta}\end{aligned} \tag{4.1}$$

where $\overline{\Delta}_{B\otimes B} = (\mathrm{id} \otimes \Psi \otimes \mathrm{id}) \circ \overline{\Delta} \otimes \overline{\Delta}$ is the coproduct in the braided tensor product coalgebra relating to $\overline{\Delta}$. We call $\overline{\Delta}$ braided comultiplication and $\underline{\Delta}$, braided coaddition.

Proposition 1. The braided (matrix) bialgebra $(B(R), \overline{\Delta}, \overline{\varepsilon})$ defined by (3.1a), (3.2a), and (3.3a) together with another Hopf algebra structure $(B(R), \underline{\Delta}, \underline{\varepsilon}, \underline{S})$ defined by (3.1a), (3.4a), (3.5a), and (3.6a) forms a braided ring.

Proof. The braided (matrix) bialgebra $B(R)$ with the above braided comultiplication $\overline{\Delta}$ and braided condition $\underline{\Delta}$ were introduced by Majid (1991, 1993) and Meyer (1995), respectively. To prove Proposition 1, we have to prove the codistributivity condition (4.1). On the generators they hold trivially. On products of the generators, for the first condition of (4.1), from the relations in $B(R)$ we have

$$\begin{aligned}&(\mathrm{id} \otimes \circ)\, \overline{\Delta}_{B\otimes B}\, \underline{\Delta}(U_1R_{12}U_2)\\ &\quad= (\mathrm{id} \otimes \circ)\, \overline{\Delta}_{B\otimes B}(U_1R_{12}U_2 \otimes 1 + U_1R_{12} \otimes U_2 + R_{21}^{-1}U_2R_{21} \otimes U_1R_{21}^{-1}\\ &\qquad + 1 \otimes U_1R_{12}U_2)\\ &\quad= (\mathrm{id} \otimes \circ)(\mathrm{id} \otimes \Psi \otimes \mathrm{id})(U_1R_{12}U_2R_{12}^{-1} \otimes U_1R_{12}U_2 \otimes 1 \otimes 1\\ &\qquad + U_1 \otimes U_1R_{12} \otimes U_2 \otimes U_2 + R_{21}^{-1}U_2 \otimes U_2R_{21} \otimes U_1 \otimes U_1R_{21}^{-1}\\ &\qquad + 1 \otimes 1 \otimes U_1R_{12}U_2R_{12}^{-1} \otimes U_1R_{12}U_2)\\ &\quad= U_1R_{12}U_2R_{12}^{-1} \otimes 1 \otimes U_1R_{12}U_2 + U_1R_{12} \otimes U_2R_{12}^{-1} \otimes U_1R_{12}U_2\\ &\qquad + R_{21}^{-1}U_2R_{21} \otimes U_1R_{21}^{-1} \otimes U_2R_{21}U_1R_{21}^{-1} + 1 \otimes U_1R_{12}U_2R_{12}^{-1} \otimes U_1R_{12}U_2\\ &\quad= (U_1R_{12}U_2R_{12}^{-1} \otimes 1 + U_1R_{12} \otimes U_2R_{12}^{-1} + R_{21}^{-1}U_2R_{21} \otimes U_1R_{21}^{-1}R_{12}^{-1}\\ &\qquad + 1 \otimes U_1R_{12}U_2R_{12}^{-1}) \otimes U_1R_{12}U_2\end{aligned}$$

On the other hand,

$$\begin{aligned}&(\underline{\Delta} \otimes \mathrm{id})\, \overline{\Delta}(U_1R_{12}U_2)\\ &\quad= (\underline{\Delta} \otimes \mathrm{id})(U_1R_{12}U_2\, R_{12}^{-1} \otimes U_1R_{12}U_2)\end{aligned}$$

$$= (U_1R_{12}U_2 \otimes 1 + U_1R_{12} \otimes U_2 + R_{21}^{-1}U_2R_{21} \otimes U_1R_{21}^{-1} + 1 \otimes U_1R_{12}U_2)R_{12}^{-1} \otimes U_1R_{12}U_2$$

The second condition and the general cases can be verified in a similar way. ■

Moreover, we find that the statement in Proposition 1 can be extended to the braided differential bialgebra $\overline{\Omega}_B$ with $\underline{\Omega}_B$.

Proposition 2. The braided differential bialgebra $(\Omega_B, \overline{\Delta}, \overline{\varepsilon})$, i.e., $\overline{\Omega}_B$ together with the braided (coadditive) Hopf algebra structure $(\Omega_B, \underline{\Delta}, \underline{\varepsilon}, \underline{S})$ given by (3.4)–(3.6) constitutes a braided ring.

Proof. Besides the calculations in the proof of Proposition 1, we need the calculations on products of other generators. For example, if we consider the product of U and $\mathrm{d}U$, by using the relations in $(\Omega_B, \overline{\Delta}, \overline{\varepsilon}, \underline{\Delta}, \underline{\varepsilon}, \underline{S})$, we have for the first condition in (4.1):

$$(\mathrm{id}\otimes\circ)\,\overline{\Delta}_{\Omega_B\otimes\Omega_B}\underline{\Delta}(U_1R_{12}\,\mathrm{d}U_2)$$

$$= (\mathrm{id}\otimes\circ)\,\overline{\Delta}_{\Omega_B\otimes\Omega_B}(U_1R_{12}\,\mathrm{d}U_2\otimes 1 + U_1R_{12}\otimes \mathrm{d}U_2 + 1\otimes U_1R_{12}\,\mathrm{d}U_2 + R_{21}^{-1}\,\mathrm{d}U_2\,R_{21}\otimes U_1R_{21}^{-1} - \lambda R_{21}^{-1}P_{12}U_1R_{12}\otimes \mathrm{d}U_2)$$

$$= (\mathrm{id}\otimes\circ)(\mathrm{id}\otimes\Psi\otimes\mathrm{id})(U_1R_{12}\,\mathrm{d}U_2\,R_{12}^{-1}\otimes U_1R_{12}U_2\otimes 1\otimes 1 + U_1R_{12}U_2R_{12}^{-1}\otimes U_1R_{12}\,\mathrm{d}U_2\otimes 1\otimes 1 + U_1\otimes U_1R_{12}\otimes \mathrm{d}U_2\otimes U_2 + U_1\otimes U_1R_{12}\otimes U_2\otimes \mathrm{d}U_2 + 1\otimes 1\otimes U_1R_{12}\,\mathrm{d}U_2\,R_{12}^{-1}\otimes U_1R_{12}U_2 + 1\otimes 1\otimes U_1R_{12}U_2R_{12}^{-1}\otimes U_1R_{12}\,\mathrm{d}U_2 + R_{21}^{-1}\,\mathrm{d}U_2\otimes U_2R_{12}\otimes U_1\otimes U_1R_{21}^{-1} + R_{21}^{-1}U_2\otimes \mathrm{d}U_2\,R_{12}\otimes U_1\otimes U_1R_{21}^{-1} - \lambda P_{12}R_{12}^{-1}U_1\otimes U_1R_{12}\otimes \mathrm{d}U_2 \otimes U_2 - \lambda P_{12}R_{12}^{-1}U_1\otimes U_1R_{12}\otimes U_2\otimes \mathrm{d}U_2)$$

$$= U_1R_{12}\,\mathrm{d}U_2\,R_{12}^{-1}\otimes 1\otimes U_1R_{12}U_2 + U_1R_{12}U_2R_{12}^{-1}\otimes 1\otimes U_1R_{12}\,\mathrm{d}U_2 + U_1R_{12}\otimes \mathrm{d}U_2\,R_{12}^{-1}\otimes U_1R_{12}U_2 + U_1R_{12}\otimes U_2R_{12}^{-1}\otimes U_1R_{12}\,\mathrm{d}U_2 + 1\otimes U_1R_{12}\,\mathrm{d}U_2\,R_{12}^{-1}\otimes U_1R_{12}U_2 + 1\otimes U_1R_{12}U_2R_{12}^{-1}\otimes U_1R_{12}\,\mathrm{d}U_2 + R_{21}^{-1}\,\mathrm{d}U_2\,R_{21}\otimes U_1R_{21}^{-1}\otimes U_2R_{21}U_1R_{21}^{-1} + R_{21}^{-1}U_2R_{21}\otimes U_1R_{21}^{-1} \otimes \mathrm{d}U_2\,R_{21}U_1R_{21}^{-1} - \lambda P_{12}R_{12}^{-1}U_1R_{12}\otimes \mathrm{d}U_2\,R_{12}^{-1}\otimes U_1R_{12}U_2 - \lambda P_{12}R_{12}^{-1}U_1R_{12}\otimes U_2R_{12}^{-1}\otimes U_1R_{12}\,\mathrm{d}U_2$$

$$= (U_1R_{12}\,\mathrm{d}U_2\,R_{12}^{-1}\otimes 1 + U_1R_{12}\otimes \mathrm{d}U_2\,R_{12}^{-1} + 1\otimes U_1R_{12}\,\mathrm{d}U_2\,R_{12}^{-1}$$

$$+ R_{21}^{-1}\,\mathrm{d}U_2\,R_{21} \otimes U_1 R_{21}^{-1} R_{12}^{-1} - \lambda P_{12} R_{12}^{-1} U_1 R_{12} \otimes \mathrm{d}U_2\,R_{12}^{-1}) \otimes U_1 R_{12} U_2$$
$$+ (U_1 R_{12} U_2 R_{12}^{-1} \otimes 1 + U_1 R_{12} \otimes U_2 R_{12}^{-1} + 1 \otimes U_1 R_{12} U_2 R_{12}^{-1}$$
$$+ R_{21}^{-1} U_2 R_{21} \otimes U_1 - \lambda P_{12} R_{12}^{-1} U_1 R_{12} \otimes U_2 R_{12}^{-1}) \otimes U_1 R_{12}\,\mathrm{d}U_2$$

On the other hand,

$$(\underline{\Delta} \otimes \mathrm{id})\,\overline{\Delta}(U_1 R_{12}\,\mathrm{d}U_2)$$
$$= (\underline{\Delta} \otimes \mathrm{id})(U_1 R_{12}\,\mathrm{d}U_2\,R_{12}^{-1} \otimes U_1 R_{12} U_2 + U_1 R_{12} U_2 R_{12}^{-1} \otimes U_1 R_{12}\,\mathrm{d}U_2)$$
$$= (U_1 R_{12}\,\mathrm{d}U_2 \otimes 1 + U_1 R_{12} \otimes \mathrm{d}U_2 + 1 \otimes U_1 R_{12}\,\mathrm{d}U_2$$
$$+ R_{21}^{-1}\,\mathrm{d}U_2\,R_{21} \otimes U_1 R_{21}^{-1} - \lambda P_{12} R_{12}^{-1} U_1 R_{12} \otimes \mathrm{d}U_2) R_{12}^{-1} \otimes U_1 R_{12} U_2$$
$$+ (U_1 R_{12} U_2 \otimes 1 + U_1 R_{12} \otimes U_2 + 1 \otimes U_1 R_{12} U_2$$
$$+ R_{21}^{-1} U_2 R_{21} \otimes U_1 R_{21}^{-1}) R_{12}^{-1} \otimes U_1 R_{12}\,\mathrm{d}U_2$$

Then the Hecke property (1.1) of R implies the first condition in (4.1) holds.

Similarly, we can verify the second condition and the general cases. ■

5. CONCLUSIONS AND DISCUSSIONS

The covariance of the braided bialgebras $\overline{\Omega}_B$, $\underline{\Omega}_B$, and Ω_{V^*} with respect to the coaction of the usual (unbraided) quantum group was considered by Iseav and Vladimirov (1995). In this paper, we extend those results to more general cases. We find that $\overline{\Omega}_B$, $\underline{\Omega}_B$ and Ω_{V^*} are also covariant under the *braided* coaction of the quantized braided group $A(R, Z)$. This extension, in general, is not trivial because the coaction here is braided. The $A(R, Z)$ covariance we obtained shows that the braided bialgebras $\overline{\Omega}_B$, $\underline{\Omega}_B$, and Ω_{V^*}, etc., have more and richer symmetries than previously expected; the quantum group covariance can be given as a special case of the $A(R, Z)$ covariance (corresponding to $Z = I$). Moreover, we also show that the braided bialgebra $(B(R), \overline{\Delta}, \overline{\varepsilon}; \underline{\Delta}, \underline{\varepsilon}, \underline{S})$ and the braided differential bialgebra $(\Omega_B, \overline{\Delta}, \overline{\varepsilon}; \underline{\Delta}, \underline{\varepsilon}, \underline{S})$ both have the braided ring structure. A similar, but different ring structure on the usual (unbraided) quantum group $A(R)$ was discussed by Majid (1994).

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China and the Science Foundation of the Educational Committee of Liaoning Province, China.

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